

Closed equations of the two-point functions for tensorial group field theory

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Abstract

In this paper we provide the closed equations that satisfy two-point correlation functions of the rank 3 and 4 tensorial group field theory. The formulation of the present problem extends the method used by Grosse and Wulkenhaar in [arXiv 0909.1389] to the tensor case. Ward-Takahashi identities and Schwinger-Dyson equations are combined to establish a nonlinear integral equation for the two-point functions. In the 3D case the solution of this equation is given perturbatively at second order of the coupling constant.

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1 Introduction

Random Tensor Models [1][2][3] extends Matrix Models [4] as promising candidates to understand Quantum Gravity in higher dimension, $D \geq 3$. The formulation of such models is based on a Feynman path integral generating randomly graphs representing simplicial pseudo manifolds of dimension D . The equivalent of the t'Hooft large N limit [5][6] for these Tensor Models has been recently discovered by Gurau [7][8][9]. The large N limit behaviour is a powerful tool which allows to understand the continuous limit of these models through, for instance, the study of critical exponents and phase transitions [10][11][12].

With the advent of the field theory formulation of Random Tensor Models, henceforth called Tensorial Group Field Theory (TGFT) [13]–[24], one addresses several different questions such as Renormalizability (for removing divergences) and the study UV behaviour of these models. It turns out that Renormalization can be consistently defined for TGFTs and most of them, for the higher rank $D \geq 3$ are UV asymptotically free [21]. This is of course very encouraging for the Geomeogenesis Scenario [15][23][24].

It becomes more and more convincing that Random Tensors and TGFT's will take a growing role for giving answers for the Quantum Gravity conundrum. Despite all these

results, a lot of questions (both conceptual and technical) arise in this framework for obtaining a final and emergent theory of General Relativity [3]. Among other goals, it would be strongly desirable to establish more connections with other studies and important results around Gravity. This is the purpose of this paper which provides the first glimpses of the extension of the recent full resolution of the correlation functions in the Grosse-Wulkenhaar (GW) model [28][29] [30].

One of the main purpose of a field theory is to find the exact value of the Green's functions also called correlation functions. Obviously, this can be a highly nontrivial task. In almost scarce cases where this is successfully done, one calls the model exactly solvable. In a recent work, the renormalizable noncommutative scalar field theory called the GW model was solved [29]–[45]. This particular noncommutative field theory projects on a matrix model and then can be seen a model for QG in $2D$. Let us review this model arising in Noncommutative Geometry. Grosse and Wulkenhaar modified the propagator of the noncommutative field theory by adding a harmonic term and showed that the resulting functional action is renormalizable at all orders of perturbation. The proof of this claim was given using the matrix basis dual to the Moyal space of functions. In [35][36][37] a new proof of the renormalizability was given in direct space using multiscale analysis [38]. The GW propagator breaks the $U(N)$ symmetry invariance in the infrared regime, but is asymptotically safe in the ultraviolet regime [39][40][41]. The model is also non invariant under translation and rotation of spacetime. The only known invariance satisfied by the model is the so-called Langman Szabo duality [42]. At the perturbative level, the associated Feynman graphs are ribbon graphs. In a recent remarkable contribution, Grosse and Wulkenhaar solve successfully all correlators in this model. Using both Ward-Takahashi identities and the Schwinger-Dyson equation, these authors provide, via Hilbert transform, a nonlinear integral equation for the two-point functions [30]. From this result, they were able to generate solutions for all correlators. Thus, the GW model is exactly nonperturbatively solvable. The question is whether or not this method may apply to other models, in particular to TGFTs dealing with higher rank tensors. We give a partial positive answer of this question. Indeed, as we will show in the following, the resolution method can be applied to find nonlinear equations for the correlations here as well. Due to the highly nontrivial equations and combinatorics, the full resolution of all correlators deserves more work which should be addressed elsewhere.

The present paper is organized as follows. In the section 2, we derive the Ward-Takahashi identities of arbitrary rank D TGFT. In section 3 we give the closed equation of the two-point correlation functions for the rank 3 TGFT. We also give the solution of this equation at second order of perturbation. In section 4 we provide the closed equation of rank 4 tensor field. We give a summary of our results and outlook of the paper in section 5.

2 Ward-Takahashi identities for arbitrary D -tensor field model

TGFT's are generally defined by an action $S[\bar{\varphi}, \varphi]$, that depends on the field φ and its conjugate $\bar{\varphi}$ defined on the compact Lie group G i.e. $\varphi : G^D \longrightarrow \mathbb{C}$; $(g_1, \dots, g_D) \longmapsto \varphi(g_1, \dots, g_D)$. For simplicity, we will always consider $G = U(1)$. We are using the Fourier transformation of the field and are defining the momentum variable associated to the group el-

ements $[g] = (g_1, g_2, \dots, g_D) \in U(1)^D$ as $[p] = (p_1, p_2, \dots, p_D) \in \mathbb{Z}^D$. Using the parametrization $g_k = e^{i\theta_k}$ we write

$$\varphi(g_1, \dots, g_D) = \sum_{p_i \in \mathbb{Z}} \varphi(p_1, \dots, p_D) e^{i \sum_k \theta_k p_k}, \quad \theta_i \in [0, 2\pi). \quad (1)$$

The Fourier transform of the field φ is denoted by $\varphi_{12\dots D} =: \varphi(p_1, \dots, p_D) =: \varphi_{[D]}$ for simplicity. The functional action $S[\bar{\varphi}, \varphi]$ is written in general case as

$$S[\bar{\varphi}, \varphi] = \sum_{p_i} \bar{\varphi}_{12\dots D} C^{-1}(p_1, p_2, \dots, p_D; p'_1, p'_2, \dots, p'_D) \varphi_{12\dots D} \prod_{i=1}^D \delta_{p_i p'_i} + S^{\text{int}} \quad (2)$$

where C stands for the propagator and S^{int} collects all vertex contributions of the interaction. Let $d\mu_C$ be the field measure associated with the covariance C , we have the relation

$$C([p]; [p']) = \int d\mu_C \varphi_{[p]} \bar{\varphi}_{[p']}, \quad d\mu_C = \prod_{[p]} d\bar{\varphi}_{[p]} d\varphi_{[p]} e^{-\bar{\varphi}_{[p]} C^{-1}([p], [p]) \varphi_{[p]}}. \quad (3)$$

The Green's functions or N -point correlation functions are defined by the relation

$$G([p]_1, [p]_2, \dots, [p]_N) = \frac{1}{\mathcal{Z}} \int d\mu_C \varphi_{[p]_1} \bar{\varphi}_{[p]_1} \dots \varphi_{[p]_N} \bar{\varphi}_{[p]_N} e^{-S^{\text{int}}}, \quad (4)$$

where \mathcal{Z} is the normalization factor also called partition function given by

$$\mathcal{Z} = \int d\mu_C e^{-S^{\text{int}}}. \quad (5)$$

Let us write the interaction term of the action (2) as $S^{\text{int}} = \lambda V[\bar{\varphi}, \varphi] =: \sum_k \lambda_k V_k[\bar{\varphi}, \varphi]$. The main idea of the perturbative theory is to expand the Green's functions as

$$\begin{aligned} G([p]_1, [p]_2, \dots, [p]_N) &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d\mu_C V^n[\bar{\varphi}, \varphi] \varphi_{[p]_1} \bar{\varphi}_{[p]_1} \dots \varphi_{[p]_N} \bar{\varphi}_{[p]_N} \\ &= \sum_{n=0}^{\infty} \lambda^n G_N^{(n)}. \end{aligned} \quad (6)$$

Using this formula, the Green's functions can be computed order by order using Dyson's theorem.

We consider the rank D tensor field φ and its conjugate $\bar{\varphi}$, which are transformed under the tensor product of D fundamental representations of the unitary group $\mathcal{U}_{\otimes}^{N_D} := \otimes_{i=1}^D U(N_i)$. Let $U^{(a)} \in U(N_a)$, $a = 1, 2, \dots, D$. The field φ and its conjugate $\bar{\varphi}$ are transformed under $U(N_a)$ as

$$\varphi_{12\dots D} \rightarrow [U^{(a)} \varphi]_{12\dots a\dots D} = \sum_{p'_a \in \mathbb{Z}} U_{p_a p'_a}^{(a)} \varphi_{12\dots a'\dots D}, \quad (7)$$

$$\bar{\varphi}_{12\dots D} \rightarrow [\bar{\varphi} U^{\dagger(a)}]_{12\dots a\dots D} = \sum_{p'_a \in \mathbb{Z}} \bar{U}_{p_a p'_a}^{(a)} \bar{\varphi}_{12\dots a'\dots D}. \quad (8)$$

p'_a or simply a' is the momentum index at the position a in the expression $\varphi_{12\dots a'\dots D}$. The kinetic action in (2) is re-expressed as follows

$$S^{\text{kin}}[\bar{\varphi}, \varphi] = \sum_{p_1, \dots, p_D} \varphi_{12\dots D} M_{12\dots D} \bar{\varphi}_{12\dots D}, \quad M_{12\dots D} = C_{12\dots D}^{-1}. \quad (9)$$

$M_{12\dots D}$ is the inverse of propagator associated to the model. Rank D tensor fields are represented by half lines made with D segments called strands. A propagator is a D stranded line and as usual connects vertices. The variation of the action S^{kin} under infinitesimal $U(N_a)$ transformation is given by

$$\delta^{(a)}[S^{\text{kin}}] = -i \sum_{p_1, \dots, p_D} \left[M \left(\varphi [\bar{B}\bar{\varphi}]^{(a)} - [B\varphi]^{(a)} \bar{\varphi} \right) \right]_{12\dots D} \quad (10)$$

where B is the infinitesimal Hermitian operator corresponding to the generator of unitary group $U(N_a)$ i.e.

$$U_{pp'}^{(a)} = \delta_{pp'}^{(a)} + iB_{pp'}^{(a)} + O(B^2), \quad \bar{U}_{pp'}^{(a)} = \delta_{pp'}^{(a)} - i\bar{B}_{pp'}^{(a)} + O(\bar{B}^2), \quad (11)$$

with $\bar{B}_{pp'}^{(a)} = B_{p'p}^{(a)}$. Consider now the theory defined with external source $F[\varphi, \bar{\varphi}; \eta, \bar{\eta}]$ as

$$F[\eta, \bar{\eta}] = \sum_{p_1, \dots, p_D} \bar{\varphi}_{12\dots D} \eta_{12\dots D} + \bar{\eta}_{12\dots D} \varphi_{12\dots D}. \quad (12)$$

The partition function of the model is re-expressed as

$$\mathcal{Z}[\eta, \bar{\eta}] = \int d\varphi d\bar{\varphi} e^{-S[\varphi, \bar{\varphi}] + F[\varphi, \bar{\varphi}; \eta, \bar{\eta}]}. \quad (13)$$

Under $U(N_a)$ infinitesimal transformation

$$\delta^{(a)}[F] = i \sum_{p_1, \dots, p_D} \left[\bar{\eta} [B\varphi]^{(a)} - [\bar{B}\bar{\varphi}]^{(a)} \eta \right]_{12\dots D}. \quad (14)$$

Let $\delta^{(\otimes)}$ be the total variation under the action of the group element $U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(D)} \in \mathcal{U}_{\otimes}^{N_D}$. Then we get the following proposition

Proposition 1. *The kinetic term of the action (2), i.e. S^{kin} and F are respectively transformed linearly as*

$$\delta^{(\otimes)} S^{\text{kin}} = \sum_{a=1}^D \delta^{(a)} S^{\text{kin}}, \quad \delta^{(\otimes)} F = \sum_{a=1}^D \delta^{(a)} F. \quad (15)$$

Then $\delta^{(\otimes)} S = 0$ if and only if $\delta^{(a)} S = 0$ for all functional quantity S , which depends on φ , $\bar{\varphi}$, η and $\bar{\eta}$.

We assume that $N_i = N$, $i = 1, 2, \dots, D$, and we take the interaction terms such that there are invariant under the transformation $U^{(a)}$ i.e. $\delta^{(a)} S^{\text{int}} = 0$. This is the new input in TGFT's: the $U(N_a)$ tensor invariance must be the one defining the interaction [1]. Note that the measure $d\varphi d\bar{\varphi}$ is also invariant under $U^{(a)}$. The variation of the partition function can be performed for $a = 1$ and the results for all value of $a \in \{1, 2, \dots, D\}$ may be deduced using proposition (1). We write

$$\begin{aligned} \frac{\delta^{(1)} \ln \mathcal{Z}[\eta, \bar{\eta}]}{\delta B_{p_m p_n}} &= \frac{1}{\mathcal{Z}[\eta, \bar{\eta}]} \int d\varphi d\bar{\varphi} \left\{ i \sum_{p_2, \dots, p_D} \left(M_{n 2 \dots D} \varphi_{n 2 \dots D} \bar{\varphi}_{m 2 \dots D} - M_{m 2 \dots D} \bar{\varphi}_{m 2 \dots D} \varphi_{n 2 \dots D} \right) \right. \\ &\quad \left. + i \sum_{p_2, \dots, p_D} \left(\bar{\eta}_{m 2 \dots D} \varphi_{n 2 \dots D} - \bar{\varphi}_{m 2 \dots D} \eta_{n 2 \dots D} \right) \right\} e^{-S[\varphi, \bar{\varphi}] + F[\varphi, \bar{\varphi}; \eta, \bar{\eta}]} = 0. \end{aligned} \quad (16)$$

Now take $\partial_{\bar{\eta}} \partial_{\eta}$ of the above expression, we get only the connected components of the correlation functions as

$$\begin{aligned} &\sum_{[p]} \left(M_{m 2 \dots D} - M_{n 2 \dots D} \right) \left\langle \left[\frac{\partial(\bar{\eta} \varphi)}{\partial \bar{\eta}} \frac{\partial(\bar{\varphi} \eta)}{\partial \eta} \right] \varphi_{n 2 \dots D} \bar{\varphi}_{m 2 \dots D} \right\rangle_c \\ &= \sum_{[p]} \left\langle \frac{\partial(\bar{\eta}_{m 2 \dots D} \varphi_{n 2 \dots D})}{\partial \bar{\eta}} \left[\frac{\partial(\bar{\varphi} \eta)}{\partial \eta} \right] - \frac{\partial(\bar{\varphi}_{m 2 \dots D} \eta_{n 2 \dots D})}{\partial \eta} \left[\frac{\partial(\bar{\eta} \varphi)}{\partial \bar{\eta}} \right] \right\rangle_c, \end{aligned} \quad (17)$$

which can be simply written as

$$\begin{aligned} &\sum_{[p]} \left(M_m - M_n \right) \left\langle \left[\frac{\partial(\bar{\eta} \varphi)}{\partial \bar{\eta}} \frac{\partial(\bar{\varphi} \eta)}{\partial \eta} \right] \varphi_n \bar{\varphi}_m \right\rangle_c \\ &= \sum_{[p]} \left\langle \frac{\partial(\bar{\eta}_m \varphi_n)}{\partial \bar{\eta}} \frac{\partial(\bar{\varphi} \eta)}{\partial \eta} \right\rangle_c - \sum_{[p]} \left\langle \frac{\partial(\bar{\varphi}_m \eta_n)}{\partial \eta} \frac{\partial(\bar{\eta} \varphi)}{\partial \bar{\eta}} \right\rangle_c. \end{aligned} \quad (18)$$

Note that the equation (18) is valid for all positions indices $a = 1, 2, \dots, D$. Let us also remark that for $m = n$ the left hand side (lhs) of the equation (18) vanishes. In the double derivative $\partial_{\bar{\eta}} \partial_{\eta}$, we fix the indices such that $\bar{\eta}_{[\alpha]} \eta_{[\beta]}$. Then comes the following proposition:

Proposition 2. *For index $a = 1$ (corresponding to $U^{(1)}$), we get the Ward-Takahashi identity*

$$\begin{aligned} &\sum_{p_2, \dots, p_D} \left(M_{m 2 \dots D} - M_{n 2 \dots D} \right) \langle \varphi_{[\alpha]} \bar{\varphi}_{[\beta]} \varphi_{n 2 \dots D} \bar{\varphi}_{m 2 \dots D} \rangle_c \\ &= \delta_{m \alpha_1} \langle \varphi_{n \alpha_2 \dots \alpha_D} \bar{\varphi}_{\beta_1 \dots \beta_D} \rangle_c - \delta_{n \beta_1} \langle \bar{\varphi}_{m \beta_2 \dots \beta_D} \varphi_{\alpha_1 \dots \alpha_D} \rangle_c, \end{aligned} \quad (19)$$

which can be re-expressed for arbitrary position a taking any value in $\{1, 2, \dots, D\}$ as

$$(M_m - M_n) \langle [\varphi_m \bar{\varphi}_n] \varphi_n \bar{\varphi}_m \rangle_c = \langle \varphi_n \bar{\varphi}_n \rangle_c - \langle \bar{\varphi}_m \varphi_m \rangle_c, \quad [\varphi_m \bar{\varphi}_n] = \sum_{p_2, \dots, p_D} \varphi_{n 2 \dots D} \bar{\varphi}_{m 2 \dots D}. \quad (20)$$

We emphasize that the position taken by the indices m and n in the relation (20) are the position of the momentum index p_a used in the transformation $U^{(a)}$. In conclusion, there are exactly D Ward-Takahashi identities for the rank D TGFT's associated with this type of invariance. Note that the Ward-Takahashi identities for Boulatov model can be found

in reference [48]. The result obtained therein radically differs from the present identities found in (20). Furthermore, we mention that we are not considering the TGFT with gauge invariance condition on the fields like in the works [18][19]. We consider here the simplest the TGFT as treated in [13][14]. Most of the result of this work might be extended to this different framework with not much work since only the propagator will be modified. Thus, one expects similar Ward identities in that gauge invariant framework .

3 Two-point functions of rank 3 TGFT

In this section we consider the just renormalizable rank 3 TGFT on compact $U(1)$ group, addressed firstly in [15]. The rank 3 tensor field is defined by $\varphi : U(1)^3 \rightarrow \mathbb{C}$, and we expand in Fourier modes as

$$\varphi(g_1, g_2, g_3) = \sum_{p_j \in \mathbb{Z}} \varphi_{123} e^{ip_1 \theta_1} e^{ip_2 \theta_2} e^{ip_3 \theta_3}, \quad \theta_i \in [0, 2\pi). \quad (21)$$

We write as usual $\varphi_{123} := \varphi_{p_1 p_2 p_3}$. The renormalizable $3D$ tensor model is defined by the action S_{3D} , in which the kinetic term take's the form

$$S_{3D}^{\text{kin}} = \sum_{[p]} \bar{\varphi}_{123} C_{123}^{-1} \varphi_{123}, \quad (22)$$

where C_{123} is the propagator. We write the resulting action for the bare quantities which involves the bare mass m_{bar} and the three wave functions renormalizations $Z_{\rho=1,2,3}$, each of which is associated with a strand index $a = 1, 2, 3$. The field strength can be modified as follows:

$$\varphi \longrightarrow (Z_1 Z_2 Z_3)^{\frac{1}{6}} \varphi = Z^{1/2} \varphi, \quad Z_\rho = 1 - \partial_{b_\rho} \Gamma_{b_1 b_2 b_3} \Big|_{b_{1,2,3}=0}, \quad (23)$$

where $\Gamma_{b_1 b_2 b_3}$ is the self-energy or one particle irreducible (1PI) two-point functions. Then, the renormalized propagator takes the form

$$C_{abc} = Z^{-1}(|a| + |b| + |c| + m^2)^{-1}, \quad a, b, c \in \mathbb{Z}. \quad (24)$$

m is the renormalized mass parameter. The interaction of the model is defined by the three contributions V_1 , V_2 , and V_3 expressed in momentum space as

$$\begin{aligned} S_{3D}^{\text{int}} &= \lambda_1 Z^2 \sum_{\substack{1,2,3 \\ 1',2',3'}} \varphi_{123} \bar{\varphi}_{321'} \varphi_{1'2'3'} \bar{\varphi}_{3'2'1} + \lambda_2 Z^2 \sum_{\substack{1,2,3 \\ 1',2',3'}} \varphi_{123} \bar{\varphi}_{32'1} \varphi_{1'2'3'} \bar{\varphi}_{3'21'} \\ &+ \lambda_3 Z^2 \sum_{\substack{1,2,3 \\ 1',2',3'}} \varphi_{123} \bar{\varphi}_{3'21} \varphi_{1'2'3'} \bar{\varphi}_{32'1'} = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3, \end{aligned} \quad (25)$$

and are represented in the figure 1.

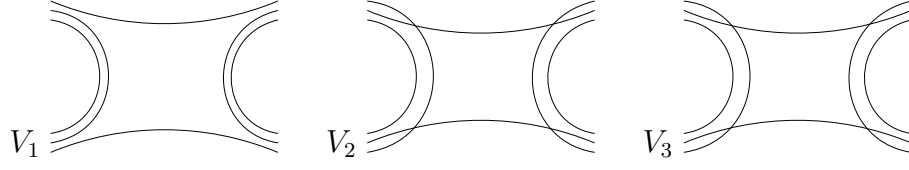


Figure 1: The vertices of rank 3 tensor model

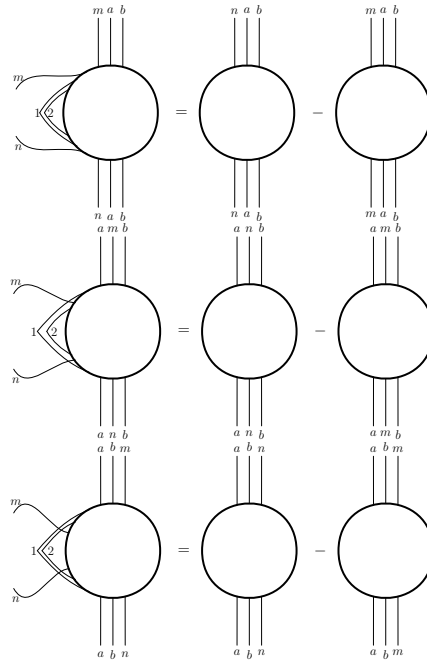


Figure 2: Ward-Takahashi identities

The Ward-Takahashi identities (20) now find the form after reducing some constraints

$$\sum_{p_2, p_3} (M_{m23} - M_{n23}) \langle \varphi_{m23} \bar{\varphi}_{n23} \varphi_{nab} \bar{\varphi}_{mab} \rangle_c = \langle \varphi_{nab} \bar{\varphi}_{nab} \rangle_c - \langle \bar{\varphi}_{mab} \varphi_{mab} \rangle_c \quad (26)$$

$$\sum_{p_1, p_3} (M_{1m3} - M_{1n3}) \langle \varphi_{1m3} \bar{\varphi}_{1n3} \varphi_{anb} \bar{\varphi}_{amb} \rangle_c = \langle \varphi_{anb} \bar{\varphi}_{anb} \rangle_c - \langle \bar{\varphi}_{amb} \varphi_{amb} \rangle_c \quad (27)$$

$$\sum_{p_1, p_2} (M_{12m} - M_{12n}) \langle \varphi_{12m} \bar{\varphi}_{12n} \varphi_{abn} \bar{\varphi}_{abm} \rangle_c = \langle \varphi_{abn} \bar{\varphi}_{abn} \rangle_c - \langle \bar{\varphi}_{abm} \varphi_{abm} \rangle_c \quad (28)$$

with $M_{abc} = C_{abc}^{-1}$. Graphically the equations (26), (27) and (28) are given in figure 2. Let $G_{[mn]ab}^{ins}$ be the two-point functions with insertion (2, 3) i.e.

$$G_{[mn]ab}^{ins} = \sum_{p_2, p_3} \langle \varphi_{m23} \bar{\varphi}_{n23} \varphi_{nab} \bar{\varphi}_{mab} \rangle_c. \quad (29)$$

The rest of this section is devoted to find perturbatively, the exact value of the renormalizable two- and four-point functions. We will use the Schwinger-Dyson equation, and then combine it with Ward-Takahashi identities to yield the closed equation that satisfies the connected two- and four-point functions. The Schwinger-Dyson equation is represented graphically in figure 3. In this figure the quantities T_{abc}^ρ and Σ_{abc}^ρ for $\rho = 1, 2, 3$, are given in the figures 4 and 5.

$$\Gamma_{abc} = \begin{array}{c} a \\ b \\ c \end{array} \cdots \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} \begin{array}{c} a \\ b \\ c \end{array} = \sum_{\rho=1}^3 \left(T_{abc}^\rho + \Sigma_{abc}^\rho \right)$$

Figure 3: Schwinger-Dyson equation for 1PI two-point functions

$$T_{abc}^1 = \begin{array}{c} da \\ \vdots \\ da \end{array} \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} \begin{array}{c} a \\ b \\ c \end{array} \quad T_{abc}^2 = \begin{array}{c} da \\ \vdots \\ da \end{array} \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} \begin{array}{c} a \\ b \\ c \end{array} \quad T_{abc}^3 = \begin{array}{c} da \\ \vdots \\ da \end{array} \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} \begin{array}{c} a \\ b \\ c \end{array}$$

Figure 4:

In the figure 3 the quantity Γ_{abc} is the self-energy or 1PI two-point functions that expresses as

$$\Gamma_{abc} = \sum_{\rho=1}^3 \Gamma_{abc}^\rho, \quad \text{where} \quad \Gamma_{abc}^\rho = T_{abc}^\rho + \Sigma_{abc}^\rho. \quad (30)$$

Also, in figures 3, 4 and 5 a single circle represents a connected graph and a double circle stands for a 1PI subgraph. Let us consider now the decomposition given in figure 6. The lhs

$$\begin{aligned}
\Sigma_{abc}^1 &= \text{Diagram 1} + \text{Diagram 2} \\
\Sigma_{abc}^2 &= \text{Diagram 3} + \text{Diagram 4} \\
\Sigma_{abc}^3 &= \text{Diagram 5} + \text{Diagram 6}
\end{aligned}$$

Figure 5:

$$\text{Diagram 7} = \text{Diagram 8} + \text{Diagram 9}$$

Figure 6: Decomposition of the two-point functions with insertion: Case where $\rho = 1$

of this equation collects all connected graphs having the vertex insertion. Cutting this vertex out one gets a four-point functions, but the four-point functions can either be disconnected (first graph on the right hand side (rhs)), or connected (second graph on the rhs). The connected four-point functions must somewhere have a 1PI four-point functions as its core and then full connected two-point functions attached to its four legs. Now, multiplying this equation by G_{abc}^{-1} means on the rhs to remove in the first graph the upper (bc)-branch attached to the insertion vertex and in the second graph the (abc)-branch attached to the 1PI four-point functions. If one now sums over p and uses the fact that the newly created vertex is $\lambda_1 Z^2$ one gets precisely the function Σ_{abc}^ρ . Then the equation (30) can be written explicitly using the decomposition of figure 6 as

$$\Sigma_{abc}^1 = Z^2 \lambda_1 \sum_p G_{abc}^{-1} G_{[ap]bc}^{ins}, \quad T_{abc}^1 = Z^2 \lambda_1 \sum_{p,q} G_{apq}. \quad (31)$$

In the same manner we can obtain the decomposition of figure 7, which allows to obtain the

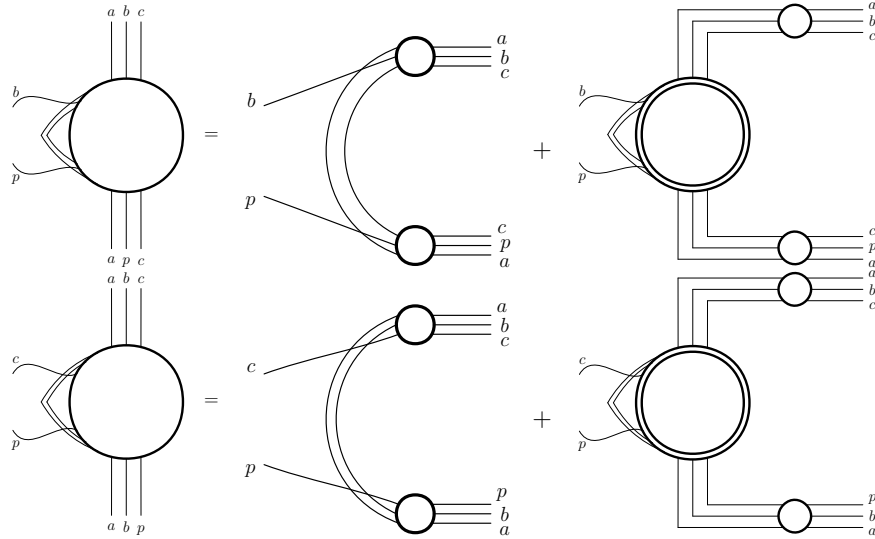


Figure 7: Decomposition of the two-point functions with insertion: Case where $\rho = 2$ and $\rho = 3$

relations

$$\Sigma_{abc}^2 = Z^2 \lambda_2 \sum_p G_{abc}^{-1} G_{[bp]ca}^{ins}, \quad T_{abc}^2 = Z^2 \lambda_2 \sum_{p,q} G_{pbq} \quad (32)$$

and

$$\Sigma_{abc}^3 = Z^2 \lambda_3 \sum_p G_{abc}^{-1} G_{[cp]ab}^{ins}, \quad T_{abc}^3 = Z^2 \lambda_3 \sum_{p,q} G_{pqc}. \quad (33)$$

Therefore using the last expressions (31), (32) and (33), the 1PI two-point functions take the form

$$\Gamma_{abc} = Z^2 \lambda_1 \sum_{p,q} G_{apq} + Z^2 \lambda_2 \sum_{p,q} G_{pbq} + Z^2 \lambda_3 \sum_{p,q} G_{pqc}$$

$$\begin{aligned}
& + Z^2 \lambda_1 \sum_p G_{abc}^{-1} G_{[ap]bc}^{ins} + Z^2 \lambda_2 \sum_p G_{abc}^{-1} G_{[bp]ca}^{ins} + Z^2 \lambda_3 \sum_p G_{abc}^{-1} G_{[cp]ab}^{ins} \\
& = Z^2 \lambda_1 \sum_{p,q} G_{apq} + Z^2 \lambda_2 \sum_{p,q} G_{pbq} + Z^2 \lambda_3 \sum_{p,q} G_{pqc} + Z \lambda_1 \sum_p G_{abc}^{-1} \frac{G_{abc} - G_{pbc}}{|p| - |a|} \\
& + Z \lambda_2 \sum_p G_{abc}^{-1} \frac{G_{bca} - G_{pca}}{|p| - |b|} + Z \lambda_3 \sum_p G_{abc}^{-1} \frac{G_{cab} - G_{pab}}{|p| - |c|}. \tag{34}
\end{aligned}$$

We assume now that the function G_{abc} satisfy the condition

$$G_{abc} = G_{bca} = G_{cab} \tag{35}$$

and then, we get the following proposition:

Proposition 3. *Symmetry properties: The connected two-point functions Γ_{abc}^2 can be obtained using Γ_{abc}^1 and replace respectively $a \rightarrow b$ and $b \rightarrow c$ and $c \rightarrow a$. In the same manner Γ_{abc}^3 can be obtained using Γ_{abc}^1 and replacing respectively $a \rightarrow c$, $b \rightarrow a$ and $c \rightarrow b$.*

Now using the relation $G_{abc}^{-1} = M_{abc} - \Gamma_{abc}$, we get

$$\Gamma_{abc}^1 = Z^2 \lambda_1 \left[\sum_{pq} \frac{1}{M_{apq} - \Gamma_{apq}} + \sum_p \frac{1}{M_{pbc} - \Gamma_{pbc}} - \sum_p \frac{1}{M_{pbc} - \Gamma_{pbc}} \frac{\Gamma_{abc} - \Gamma_{pbc}}{Z(|a| - |p|)} \right], \tag{36}$$

$$\Gamma_{abc}^2 = Z^2 \lambda_2 \left[\sum_{pq} \frac{1}{M_{pbq} - \Gamma_{pbq}} + \sum_p \frac{1}{M_{pca} - \Gamma_{pca}} - \sum_p \frac{1}{M_{pca} - \Gamma_{pca}} \frac{\Gamma_{bca} - \Gamma_{pca}}{Z(|b| - |p|)} \right], \tag{37}$$

$$\Gamma_{abc}^3 = Z^2 \lambda_3 \left[\sum_{pq} \frac{1}{M_{pqc} - \Gamma_{pqc}} + \sum_p \frac{1}{M_{pab} - \Gamma_{pab}} - \sum_p \frac{1}{M_{pab} - \Gamma_{pab}} \frac{\Gamma_{cab} - \Gamma_{pab}}{Z(|c| - |p|)} \right]. \tag{38}$$

For the rest of this section we consider the connected two-point functions Γ_{abc}^1 and finally Γ_{abc}^2 and Γ_{abc}^3 will be deduced using the proposition (3). Then we pass to renormalized quantities using the Taylor expansion as

$$\Gamma_{abc}^1 = Z M_{abc}^{bar} - M_{abc}^{phys} + \Gamma_{abc}^{phys}, \quad \Gamma_{000}^{phys} = 0 = \partial \Gamma_{000}^{phys} \tag{39}$$

such that

$$M_{abc}^{phys} = |a| + |b| + |c| + m^2, \quad M_{abc}^{bar} = |a| + |b| + |c| + m_{bar}^2. \tag{40}$$

We get after replacing the expression of M_{abc} ,

$$\Gamma_{abc}^1 = (Z - 1)(|a| + |b| + |c|) + Z m_{bar}^2 - m^2 + \Gamma_{abc}^{phys}, \tag{41}$$

which expresses the relation between renormalized and bare quantities. The equation (34) takes the form (we set $\lambda_1 = \lambda$)

$$Z m_{bar}^2 - m^2 + (Z - 1)(|a| + |b| + |c|) + \Gamma_{abc}^{phys} = Z^2 \lambda \sum_{p,q} \frac{1}{|p| + |q| + |a| + m^2 - \Gamma_{pqa}^{phys}}$$

$$+Z\lambda\left[\sum_p\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}-\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}\frac{\Gamma_{abc}^{phys}-\Gamma_{pbc}^{phys}}{(|a|-|p|)}\right].(42)$$

For $a = b = c = 0$ the relation of mass variation after renormalization is written as

$$\begin{aligned} Zm_{bar}^2 - m^2 &= Z^2\lambda\sum_{p,q}\frac{1}{|p|+|q|+m^2-\Gamma_{pq0}^{phys}} + Z\lambda\sum_p\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}} \\ &- Z\lambda\sum_p\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}}\frac{\Gamma_{p00}^{phys}}{|p|}. \end{aligned} \quad (43)$$

Inserting the equation (43) in (42), we get the closed equation of the two-point functions of renormalizable rank 3 TGFT as

$$\begin{aligned} (Z-1)(|a|+|b|+|c|)+\Gamma_{abc}^{phys} &= Z^2\lambda\sum_{p,q}\left[\frac{1}{|p|+|q|+|a|+m^2-\Gamma_{pqa}^{phys}}-\frac{1}{|p|+|q|+m^2-\Gamma_{pqa}^{phys}}\right] \\ &+ Z\lambda\sum_p\left[\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}-\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}\frac{\Gamma_{abc}^{phys}-\Gamma_{pbc}^{phys}}{(|a|-|p|)}\right. \\ &\quad \left.-\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}}+\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}}\frac{\Gamma_{p00}^{phys}}{|p|}\right]. \end{aligned} \quad (44)$$

The equation (44) is still very complicated compared to an equivalent one in [28]. To simplify it and get explicit solution, we pass to the integral transforms. The process is to set

$$\sum_{p\in\mathbb{Z}} = 2\int_0^\infty d|p|, \quad \sum_{p,q\in\mathbb{Z}} = 2\int_0^\infty |p|d|p|. \quad (45)$$

We also assume that $\Gamma_{abc} = \Gamma_{|a||b||c|}$. Then we get the integral equation of (44) as

$$\begin{aligned} &(Z-1)(|a|+|b|+|c|)+\Gamma_{abc}^{phys} \\ &= 2Z^2\lambda\int_0^\infty |p|d|p|\left[\frac{1}{2|p|+|a|+m^2-\Gamma_{ppa}^{phys}}-\frac{1}{2|p|+m^2-\Gamma_{pp0}^{phys}}\right] \\ &+ 2Z\lambda\int_0^\infty d|p|\left[\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}-\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}}\right. \\ &\quad \left.-\frac{1}{|p|+|b|+|c|+m^2-\Gamma_{pbc}^{phys}}\frac{\Gamma_{abc}^{phys}-\Gamma_{pbc}^{phys}}{(|a|-|p|)}+\frac{1}{|p|+m^2-\Gamma_{p00}^{phys}}\frac{\Gamma_{p00}^{phys}}{|p|}\right] \end{aligned} \quad (46)$$

with $p \in \mathbb{R}^+$. We introduce a change of variables

$$|a| = m^2\frac{\alpha}{1-\alpha}, \quad |b| = m^2\frac{\beta}{1-\beta}, \quad |c| = m^2\frac{\gamma}{1-\gamma}, \quad |p| = m^2\frac{\rho}{1-\rho}, \quad (47)$$

$$\Gamma_{abc}^{phys} = m^2\frac{\Gamma_{\alpha\beta\gamma}}{(1-\alpha)(1-\beta)(1-\gamma)}. \quad (48)$$

We also take the cutoff Λ such that $p_\Lambda = m^2 \frac{\Lambda}{1-\Lambda}$. Let us now define the quantity $G_{\alpha\beta\gamma}$ as

$$1 - \alpha\beta - \alpha\gamma - \beta\gamma + 2\alpha\beta\gamma - \Gamma_{\alpha\beta\gamma} = \frac{1 - \alpha\beta - \alpha\gamma - \beta\gamma + 2\alpha\beta\gamma}{G_{\alpha\beta\gamma}}. \quad (49)$$

Let $\mathcal{J}_{\alpha\beta\gamma}$, $\mathcal{L}_{\alpha\beta\gamma}$ and \mathcal{K}_α are three integrals relation given by

$$\mathcal{J}_{\alpha\beta\gamma} = \int_0^1 \frac{d\rho}{(\alpha - \rho)} \frac{G_{\rho\beta\gamma}}{(1 - \beta\rho - \gamma\rho - \gamma\beta + 2\rho\gamma\beta)}, \quad (50)$$

$$\mathcal{L}_{\alpha\beta\gamma} = \int_0^1 \frac{d\rho}{(1 - \rho)} \frac{G_{\rho\beta\gamma} - 1}{(\alpha - \rho)}, \quad (51)$$

$$\mathcal{K}_\alpha = m^2 \frac{\int_0^1 \frac{\rho d\rho}{(1-\rho)} \left(\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2} \right)}{1 + \frac{2\lambda}{m^2} \int_0^1 d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right)}. \quad (52)$$

Then we get the following theorem

Theorem 1. *The connected two-point functions $G_{\alpha\beta\gamma}$ of the renormalizable rank 3 TGFT on $U(1)$ satisfies the closed integral equation*

$$\begin{aligned} G_{\alpha\beta\gamma} = & 1 + \lambda' \left\{ \mathcal{Y} + \int_0^1 d\rho G_{\rho 00} + (1 - \alpha)(1 - \beta)(1 - \gamma)\mathcal{J}_{\alpha\beta\gamma} \right. \\ & + \frac{(1 - \alpha)(1 - \beta)(1 - \gamma)G_{\alpha\beta\gamma}}{1 - \alpha\beta - \alpha\gamma - \beta\gamma + 2\alpha\beta\gamma} \left[-\mathcal{Y} - \int_0^1 d\rho G_{\rho 00} + \mathcal{K}_\alpha \right. \\ & \left. \left. + \int_0^1 d\rho \frac{G_{\rho\beta\gamma} - G_{\rho 00}}{1 - \rho} - \int_0^1 d\rho \frac{G_{\rho\beta\gamma}}{\alpha - \rho} + (1 - \alpha)\mathcal{L}_{\alpha\beta\gamma} - \mathcal{L}_{000} \right] \right\} \end{aligned} \quad (53)$$

where

$$\mathcal{Y} = \lim_{\epsilon \rightarrow 0} \int_0^1 d\rho \frac{G_{\rho\epsilon 0} - G_{\rho 00}}{\epsilon\rho}, \quad \lambda' = \frac{2\lambda}{m^2}. \quad (54)$$

Proof. Using the transformations given in the equations (47) and (48), the expression (46) takes the form

$$\begin{aligned} & (Z - 1) \left(\frac{\alpha}{1 - \alpha} + \frac{\beta}{1 - \beta} + \frac{\gamma}{1 - \gamma} \right) + \frac{\Gamma_{\alpha\beta\gamma}}{(1 - \alpha)(1 - \beta)(1 - \gamma)} \\ & = 2Z^2\lambda \int_0^\Lambda \frac{\rho d\rho}{(1 - \rho)} \left[\frac{(1 - \alpha)}{1 - \rho^2 - 2\alpha\rho + 2\alpha\rho^2 - \Gamma_{\rho\rho\alpha}} - \frac{1}{1 - \rho^2 - \Gamma_{\rho\rho 0}} \right] \\ & + \frac{2Z\lambda}{m^2} \int_0^\Lambda \frac{d\rho}{(1 - \rho)} \left[\frac{(1 - \beta)(1 - \gamma)}{1 - \beta\rho - \gamma\rho - \gamma\beta + 2\rho\gamma\beta - \Gamma_{\rho\beta\gamma}} - \frac{1}{1 - \Gamma_{\rho 00}} \right. \\ & \left. - \frac{1}{1 - \beta\rho - \gamma\rho - \gamma\beta + 2\rho\gamma\beta - \Gamma_{\rho\beta\gamma}} \frac{(1 - \rho)\Gamma_{\alpha\beta\gamma} - (1 - \alpha)\Gamma_{\rho\beta\gamma}}{\alpha - \rho} \right. \\ & \left. + \frac{1}{1 - \Gamma_{\rho 00}} \frac{\Gamma_{\rho 00}}{\rho} \right]. \end{aligned} \quad (55)$$

Noting that β and γ are symmetric parameters in the equation (55). This implies that $\Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha\gamma\beta}$. Let us now take $\frac{\partial}{\partial\alpha}\Big|_{\alpha=\beta=\gamma=0}$ and $\frac{\partial}{\partial\beta}\Big|_{\alpha=\beta=\gamma=0}$ of the above equation. We come to the relations that satisfies the renormalized wave function Z :

$$Z - 1 = 2Z^2\lambda \int_0^\Lambda \frac{\rho d\rho}{(1-\rho)} \frac{(-1 + 2\rho - \rho^2 + \Gamma'_{\rho\rho 0} + \Gamma_{\rho\rho 0})}{(1 - \rho^2 - \Gamma_{\rho\rho 0})^2} - \frac{2Z\lambda}{m^2} \int_0^\Lambda d\rho \frac{\Gamma_{\rho 00}}{\rho^2(1 - \Gamma_{\rho 00})}, \quad (56)$$

and

$$Z - 1 = \frac{2Z\lambda}{m^2} \int_0^\Lambda \frac{d\rho}{(1-\rho)} \left[\frac{-1 + \rho + \Gamma_{\rho 00} + \Gamma'_{\rho 00}}{(1 - \Gamma_{\rho 00})^2} - \frac{(\rho + \Gamma'_{\rho 00})\Gamma_{\rho 00}}{\rho(1 - \Gamma_{\rho 00})^2} - \frac{\Gamma'_{\rho 00}}{\rho(1 - \Gamma_{\rho 00})} \right], \quad (57)$$

where we take $\Gamma'_{\rho 00} =: \frac{\partial \Gamma_{\rho\beta\gamma}}{\partial\beta}\Big|_{\beta=\gamma=0}$ or $\Gamma'_{\rho 00} =: \frac{\partial \Gamma_{\rho\beta\gamma}}{\partial\gamma}\Big|_{\beta=\gamma=0}$ and $\Gamma'_{\rho\rho 0} =: \frac{\partial \Gamma_{\rho\rho\alpha}}{\partial\alpha}\Big|_{\alpha=0}$. Now let us pass to the new function $G_{\alpha\beta\gamma}$ given in (49). We find the following relations

$$\rho + \Gamma'_{\rho 00} = \frac{\rho}{G_{\rho 00}} + \frac{G'_{\rho 00}}{G_{\rho 00}^2}, \quad 2\rho - 2\rho^2 + \Gamma'_{\rho\rho 0} = \frac{2\rho(1-\rho)}{G_{\rho\rho 0}} + \frac{(1-\rho^2)G'_{\rho\rho 0}}{G_{\rho\rho 0}}. \quad (58)$$

Therefore the equation (57) reduces to

$$Z^{-1} = 1 + \frac{2\lambda}{m^2} \int_0^\Lambda d\rho \left[\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right], \quad (59)$$

and (55) takes the form

$$\begin{aligned} & ZG_{\alpha\beta\gamma} - 1 - (Z-1) \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} G_{\alpha\beta\gamma} \\ &= \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} G_{\alpha\beta\gamma} \left\{ 2Z^2\lambda \int_0^\Lambda \frac{\rho d\rho}{(1-\rho)} \left[\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2} \right] \right. \\ &+ \frac{2Z\lambda}{m^2} \int_0^\Lambda \frac{d\rho}{(1-\rho)} \left[\frac{(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} - G_{\rho 00} + \frac{(1-\alpha)(G_{\rho\beta\gamma}-1)}{(\alpha-\rho)} + \frac{G_{\rho 00}-1}{\rho} \right. \\ &\left. \left. - \frac{G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} \frac{(1-\rho)(1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma)(G_{\alpha\beta\gamma}-1)}{(\alpha-\rho)G_{\alpha\beta\gamma}} \right] \right\}. \quad (60) \end{aligned}$$

Inserting (59) into the left hand side of (60) and dividing by Z , one gets

$$\begin{aligned} G_{\alpha\beta\gamma} &= Z^{-1} - \frac{2\lambda}{m^2} \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} G_{\alpha\beta\gamma} \int_0^\Lambda d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right) \\ &- \frac{2\lambda}{m^2} \int_0^\Lambda d\rho \frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} \cdot \frac{(G_{\alpha\beta\gamma}-1)}{(\alpha-\rho)} \\ &+ \frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\alpha\beta\gamma}}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left\{ \frac{2\lambda}{Z^{-1}} \int_0^\Lambda \frac{\rho d\rho}{(1-\rho)} \left[\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2} \right] \right. \\ &+ \frac{2\lambda}{m^2} \int_0^\Lambda \frac{d\rho}{(1-\rho)} \left[\frac{(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} - G_{\rho 00} + \frac{(1-\alpha)(G_{\rho\beta\gamma}-1)}{(\alpha-\rho)} \right. \\ &\left. \left. + \frac{G_{\rho 00}-1}{\rho} \right] \right\}. \quad (61) \end{aligned}$$

Replacing (59) (61) yields

$$\begin{aligned}
G_{\alpha\beta\gamma} &= 1 + \frac{2\lambda}{m^2} \left\{ \int_0^\Lambda d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right) - \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} G_{\alpha\beta\gamma} \right. \\
&\times \int_0^\Lambda d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right) - \int_0^\Lambda d\rho \frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} \cdot \frac{(G_{\alpha\beta\gamma}-1)}{(\alpha-\rho)} \\
&+ \frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\alpha\beta\gamma}}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[m^2 \frac{\int_0^\Lambda \frac{\rho d\rho}{(1-\rho)} \left(\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2} \right)}{1 + \frac{2\lambda}{m^2} \int_0^\Lambda d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right)} \right. \\
&+ \int_0^\Lambda \frac{d\rho}{(1-\rho)} \left(\frac{(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} - G_{\rho 00} + \frac{(1-\alpha)(G_{\rho\beta\gamma}-1)}{(\alpha-\rho)} \right. \\
&\left. \left. \left. + \frac{G_{\rho 00}-1}{\rho} \right) \right] \right\}. \tag{62}
\end{aligned}$$

Simplifying identical terms we get the result of Theorem 1. \square

Note that $G_{000} = 1$ and $\partial G_{000} = 0$. The equation (62) shows the occurrence of the singular integral kernel $\int_0^\Lambda \frac{d\rho}{\rho-\alpha}$, $\int_0^\Lambda \frac{d\rho}{1-\rho}$ and $\int_0^\Lambda \frac{d\rho}{\rho}$ for $\Lambda = 1$, which needs to be removed. We will use the Cauchy principal value of the divergent integrals and also take the limit value at points 0 and 1 i.e.

$$\int_0^1 = \lim_{\epsilon \rightarrow 0} \left[\int_0^{a-\epsilon} + \int_{a+\epsilon}^1 \right], \quad a \in (0, 1), \quad \int_0^1 = \lim_{\epsilon \rightarrow 0, \epsilon' \rightarrow 1} \int_\epsilon^{\epsilon'} \tag{63}$$

The nonlinear integral equation (53) is of the form

$$G_{\alpha\beta\gamma} = 1 + \lambda \int_0^1 f(G_{\alpha\beta\gamma}, G_{\rho\beta\gamma}, G_{\rho\alpha 0}, G_{\rho 00}, \mathcal{Y}, \alpha, \beta, \gamma) d\rho. \tag{64}$$

Now we can easily see that (53) suffers for the lack of symmetry. This inconvenience is due to the position of parameter α . So taken $\alpha = 0$ we get the symmetric solution given in the following proposition

Proposition 4. *At first order in λ the solution of the equation (53) for $\alpha = 0$ is given by*

$$G_{0\beta\gamma} = 1 + \lambda' \left[1 + \frac{(1-\beta)(1-\gamma)}{1-\beta\gamma} \left(\ln \frac{1+\beta\gamma-\beta-\gamma}{1-\beta\gamma} - 1 \right) \right] = 1 + \lambda' \mathcal{K}_{0\beta\gamma}. \tag{65}$$

Then, using the symmetry properties of proposition 3 we get the symmetric solution $G_{\alpha\beta\gamma}^{sym}$ as

$$G_{\alpha\beta\gamma}^{sym} = 1 + \lambda'_1 \mathcal{K}_{0\beta\gamma} + \lambda'_2 \mathcal{K}_{0\gamma\alpha} + \lambda'_3 \mathcal{K}_{0\alpha\beta} \tag{66}$$

with $\lambda'_\rho = \frac{2\lambda_\rho}{m^2}$; $\rho = 1, 2, 3$ and $\alpha, \beta, \gamma \in [0, 1)$.

We use the symmetry relation (35) and get the following result

Theorem 2. *The closed equation of the symmetric two-point functions $G_{\alpha\beta\gamma}$ satisfies the nonlinear integral equation*

$$\begin{aligned} G_{\alpha\beta\gamma} = & 1 + \lambda' \left\{ \mathcal{Y} + \int_0^1 d\rho G_{\rho 00} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[\int_0^1 d\rho \left(\frac{G_{\rho\beta\gamma}}{\alpha-\rho} \right. \right. \right. \\ & + \left. \left. \frac{(2\beta\gamma-\beta-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} \right) + G_{\alpha\beta\gamma} \left[\int_0^1 d\rho \frac{G_{\rho\alpha 0}-G_{\rho 00}}{1-\rho} + \int_0^1 d\rho \frac{G_{\rho 00}}{\rho} - \mathcal{Y} \right. \right. \\ & \left. \left. \left. - \int_0^1 d\rho G_{\rho 00} - G_{0\alpha 0}^{-1} \left(\int_0^1 d\rho \frac{G_{\rho\alpha 0}}{\rho} + \alpha \int_0^1 d\rho \frac{G_{\rho\alpha 0}}{1-\alpha\rho} + \int_0^1 d\rho \frac{G_{\rho 00}}{\alpha-\rho} \right) \right] \right\}. \quad (67) \end{aligned}$$

Proof. Using the relation (35) we can extract the quantity \mathcal{K}_α after simplification as

$$\begin{aligned} \mathcal{K}_\alpha = & -G_{0\alpha\beta}^{-1} \left[\int_0^1 d\rho \frac{G_{\rho\alpha\beta}}{\rho} - (2\alpha\beta - \alpha - \beta) \int_0^1 d\rho \frac{G_{\rho\alpha\beta}}{1-\alpha\rho-\beta\rho-\alpha\beta+2\alpha\beta\rho} \right. \\ & \left. + \int_0^1 d\rho \frac{G_{\rho 0\beta}}{\alpha-\rho} - \beta \int_0^1 d\rho \frac{G_{\rho 0\beta}}{1-\beta\rho} \right] + \int_0^1 d\rho \frac{G_{\rho\alpha\beta}-G_{\rho 0\beta}}{1-\rho} + \int_0^1 d\rho \left(\frac{1}{\rho} + \frac{1}{\alpha-\rho} \right). \quad (68) \end{aligned}$$

Then remark that \mathcal{K}_α is function of only the parameter α . We then take $\beta = 0$ in the last equation and we get

$$\begin{aligned} \mathcal{K}_\alpha = & -G_{0\alpha 0}^{-1} \left(\int_0^1 d\rho \frac{G_{\rho\alpha 0}}{\rho} + \alpha \int_0^1 d\rho \frac{G_{\rho\alpha 0}}{1-\alpha\rho} + \int_0^1 d\rho \frac{G_{\rho 00}}{\alpha-\rho} \right) \\ & + \int_0^1 d\rho \frac{G_{\rho\alpha 0}-G_{\rho 00}}{1-\rho} + \int_0^1 d\rho \left(\frac{1}{\rho} + \frac{1}{\alpha-\rho} \right). \quad (69) \end{aligned}$$

By replacing the relation (69) in expression (62) we get the desired result. \square

Now we are reaching the point where it is possible to give the solution of the equation (67). Let us write the solution of this equation as

$$G_{\alpha\beta\gamma} = 1 + \sum_{n=1}^{\infty} (\lambda')^n X_{\alpha\beta\gamma}^{(n)} \quad (70)$$

The n order terms $X_{\alpha\beta\gamma}^{(n)}$ can be deduced by iteration. We give here the quantities $X_{\alpha\beta\gamma}^{(1)}$ and $X_{\alpha\beta\gamma}^{(2)}$ in the following statement

Proposition 5. *Pertubatively, at second order in λ the symmetry solution of the equation (67) using the Cauchy principal value is given by*

$$\begin{aligned} G_{\alpha\beta\gamma} = & 1 + \lambda' \mathcal{X}_{\alpha\beta\gamma}^{(1)} + \lambda'^2 \left\{ \frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[\mathcal{X}_{\alpha\beta\gamma}^{(1)} \left(\ln \frac{(1-\alpha)^2}{\alpha} - 1 \right) \right. \right. \\ & + \int_0^1 d\rho \frac{(2\beta\gamma-\beta-\gamma)\mathcal{X}_{\rho\beta\gamma}^{(1)}}{1-\beta\rho-\gamma\rho-\beta\gamma+2\rho\gamma\beta} + \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)}-\mathcal{X}_{\rho 00}^{(1)}}{1-\rho} - \alpha \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)}}{1-\alpha\rho} \\ & \left. \left. - \frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)}-\mathcal{X}_{\rho 00}^{(1)}+\mathcal{X}_{0\alpha 0}^{(1)}}{\rho} - \mathcal{X}_{0\alpha 0}^{(1)} \ln \frac{(1-\alpha)^2}{\alpha} \right] \right\} + O(\lambda'^3), \quad (71) \end{aligned}$$

where $G_{000} = 1$ and where the first order term $\mathcal{X}_{\alpha\beta\gamma}^{(1)}$ is

$$\mathcal{X}_{\alpha\beta\gamma}^{(1)} = 1 + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left(\ln(1-\alpha) - 1 + \ln \frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma} \right). \quad (72)$$

The exact value of the integrals in the rhs of (71) are given using the following relations

$$\begin{aligned} \int_0^1 d\rho \frac{\mathcal{X}_{\rho\beta\gamma}^{(1)}}{a-\rho} &= \ln \frac{a}{1-a} + \frac{(1-\beta)(1-\gamma)}{1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma} \left(-1 + \ln \frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma} \right) \\ &\times \left((1-a) \ln \frac{a}{1-a} + \frac{\beta\gamma-\beta-\gamma+1}{2\beta\gamma-\beta-\gamma} \ln \frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma} \right) \\ &+ \frac{(1-a)(1-\beta)(1-\gamma)}{1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma} \left(\frac{\pi^2}{6} - Li_2 \frac{-a}{1-a} + \ln(1-a) \ln \frac{a}{1-a} - \frac{1}{1-a} \right) \\ &+ \frac{(1-\beta\gamma)(1-\beta)(1-\gamma)}{(\beta+\gamma-2\beta\gamma)(1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma)} \left(\frac{\pi^2}{6} - Li_2 \frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma} \right. \\ &\left. - \ln \frac{\beta+\gamma-2\beta\gamma}{1-\beta\gamma} \ln \frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma} \right) \end{aligned} \quad (73)$$

and

$$\begin{aligned} \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)} - \mathcal{X}_{\rho 00}^{(1)} + \mathcal{X}_{0\alpha 0}^{(1)}}{\rho} &= \frac{(1-\alpha)^2}{\alpha} \ln(1-\alpha) \left(\ln(1-\alpha) - 1 \right) - (1-\alpha) \left(\frac{\pi^2}{6} - 1 \right) \\ &+ \frac{1-\alpha}{\alpha} \left(\ln \alpha \ln(1-\alpha) + Li_2(1-\alpha) - \frac{\pi^2}{6} \right) + \frac{\pi^2}{6} - 1 \end{aligned} \quad (74)$$

where

$$Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \quad Li_2(1) = \frac{\pi^2}{6}, \quad Li_2(-1) = -\frac{\pi^2}{12}, \quad Li_2(0) = 0. \quad (75)$$

Let us immediately emphasize that the above solution is related to the coupling constant λ_1 . To establish the full solution of the two-point functions of our model, which takes into account the three coupling constants λ_ρ , $\rho = 1, 2, 3$ we must use the symmetry condition of proposition 3. The end result is given by the sum of the three equations (36), (37) and (38). Therefore the two-point functions $G_{\alpha\beta\gamma}^{Sym}$ of 3D tensor model is given by the relation

$$\begin{aligned} G_{\alpha\beta\gamma}^{sym} &= 1 + \lambda'_1 \mathcal{X}_{\alpha\beta\gamma}^{(1)} + \lambda_1^2 \left\{ \frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[\mathcal{X}_{\alpha\beta\gamma}^{(1)} \left(\ln \frac{(1-\alpha)^2}{\alpha} - 1 \right) \right. \right. \\ &+ \int_0^1 d\rho \frac{(2\beta\gamma-\beta-\gamma)\mathcal{X}_{\rho\beta\gamma}^{(1)}}{1-\beta\rho-\gamma\rho-\beta\gamma+2\beta\gamma\rho} + \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)} - \mathcal{X}_{\rho 00}^{(1)}}{1-\rho} - \alpha \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)}}{1-\alpha\rho} \\ &- \frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 d\rho \frac{\mathcal{X}_{\rho\alpha 0}^{(1)} - \mathcal{X}_{\rho 00}^{(1)} + \mathcal{X}_{0\alpha 0}^{(1)}}{\rho} - \mathcal{X}_{0\alpha 0}^{(1)} \ln \frac{(1-\alpha)^2}{\alpha} \left. \right] \Big\} + O(\lambda_1^3) \\ &+ \lambda'_2 \mathcal{X}_{\beta\gamma\alpha}^{(1)} + \lambda_2^2 \left\{ \frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[\mathcal{X}_{\beta\gamma\alpha}^{(1)} \left(\ln \frac{(1-\beta)^2}{\beta} - 1 \right) \right. \right. \\ &+ \int_0^1 d\rho \frac{(2\alpha\gamma-\alpha-\gamma)\mathcal{X}_{\rho\gamma\alpha}^{(1)}}{1-\alpha\rho-\gamma\rho-\alpha\gamma+2\alpha\gamma\rho} + \int_0^1 d\rho \frac{\mathcal{X}_{\rho\beta 0}^{(1)} - \mathcal{X}_{\rho 00}^{(1)}}{1-\rho} - \beta \int_0^1 d\rho \frac{\mathcal{X}_{\rho\beta 0}^{(1)}}{1-\beta\rho} \\ &- \frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 d\rho \frac{\mathcal{X}_{\rho\beta 0}^{(1)} - \mathcal{X}_{\rho 00}^{(1)} + \mathcal{X}_{0\beta 0}^{(1)}}{\rho} - \mathcal{X}_{0\beta 0}^{(1)} \ln \frac{(1-\beta)^2}{\beta} \left. \right] \Big\} + O(\lambda_2^3) \end{aligned}$$

$$\begin{aligned}
& + \lambda'_3 \mathcal{X}_{\gamma\beta\alpha}^{(1)} + \lambda_3'^2 \left\{ \frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \left[\mathcal{X}_{\gamma\beta\alpha}^{(1)} \left(\ln \frac{(1-\gamma)^2}{\gamma} - 1 \right) \right. \right. \\
& + \int_0^1 d\rho \frac{(2\alpha\beta - \alpha - \beta) \mathcal{X}_{\rho\alpha\beta}^{(1)}}{1-\alpha\rho-\beta\rho-\alpha\beta+2\alpha\beta\rho} + \int_0^1 d\rho \frac{\mathcal{X}_{\rho\gamma 0}^{(1)} - \mathcal{X}_{\rho 0 0}^{(1)}}{1-\rho} - \gamma \int_0^1 d\rho \frac{\mathcal{X}_{\rho\gamma 0}^{(1)}}{1-\gamma\rho} \\
& \left. \left. - \frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 d\rho \frac{\mathcal{X}_{\rho\gamma 0}^{(1)} - \mathcal{X}_{\rho 0 0}^{(1)} + \mathcal{X}_{0\gamma 0}^{(1)}}{\rho} - \mathcal{X}_{0\gamma 0}^{(1)} \ln \frac{(1-\gamma)^2}{\gamma} \right] \right\} + O(\lambda_3'^3) \quad (76)
\end{aligned}$$

where $\lambda'_\rho = 2\lambda_\rho/m^2$; $\rho = 1, 2, 3$, $\alpha, \beta, \gamma \in (0, 1)$ and $G_{000} = 1$. Noting that the solution (76) satisfies the condition (35) if and only if we set $\lambda'_1 = \lambda'_2 = \lambda'_3$. Let us also emphasize that the higher order solution can be get perturbatively by iteration.

4 Closed equation for two-point functions of rank 4 TGFT

The same method use in last section will be performed here to establish the renormalized two-point functions of rank 4 tensor field firstly given in [13]. We provide the master equation of the two-point functions. The action S_{4D} of the model is also subdivided into two terms as

$$S_{4D} = S_{4D}^{\text{kin}} + S_{4D}^{\text{int}}. \quad (77)$$

The kinetic term S_{4D}^{kin} is given by

$$S_{4D}^{\text{kin}} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234} \left(\sum_{i=1}^4 p_i^2 + m^2 \right) \bar{\varphi}_{1234}. \quad (78)$$

Noting that in four dimensional case the renormalization is guaranteed by the presence of the propagator associated with the heat kernel [27]:

$$C([p]) = \left(\sum_{i=1}^4 p_i^2 + m^2 \right)^{-1} = M_{1234}^{-1}. \quad (79)$$

S_{4D}^{int} is related to the interaction, which is divided into three fundamental contributions $V_{6,1}$, $V_{6,2}$ and $V_{4,1}$ given by

$$V_{6,1} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234} \bar{\varphi}_{1'2'3'4'} \varphi_{1''2''3''4''} \bar{\varphi}_{12''3''4''} + \text{permutations} \quad (80)$$

$$V_{6,2} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234} \bar{\varphi}_{1'2'3'4'} \varphi_{1''2''3''4''} \bar{\varphi}_{12''3''4''} + \text{permutations} \quad (81)$$

$$V_{4,1} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234} \bar{\varphi}_{1'2'3'4'} \varphi_{12'3'4'} + \text{permutations} \quad (82)$$

and an anomalous term, namely $V_{4,2}$

$$V_{4,2} = \left(\sum_{p_j \in \mathbb{Z}} \bar{\varphi}_{1234} \varphi_{1234} \right) \left(\sum_{p_j \in \mathbb{Z}} \bar{\varphi}_{1'2'3'4'} \varphi_{1'2'3'4'} \right). \quad (83)$$

$$\Gamma_{abcd}^{6,1} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + 3 \text{ Permutation } \rho$$

Figure 10:

$$\Gamma_{abcd}^{6,2} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + 5 \text{ Permutation } \rho\rho'$$

Figure 11:

$$\Gamma_{abcd}^{4,1} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + 3 \text{ Permutation } \rho$$

Figure 12:

over two is $T_{abcd}^{4,1}$. Then the relations given in figures 10, 11 and 12 are re-expressed simply as

$$\Gamma_{abcd}^{6,1} = \sum_{\rho} \Gamma_{abcd}^{6,1;\rho}, \quad \Gamma_{abcd}^{6,2} = \sum_{\rho\rho'} \Gamma_{abcd}^{6,2;\rho\rho'}, \quad \Gamma_{abcd}^{4,1} = \sum_{\rho} \Gamma_{abcd}^{4,1;\rho} \quad (88)$$

with

$$\Gamma_{abcd}^{6,1;\rho} = T_{abcd}^{6,1;\rho} + \Sigma_{abcd}^{6,1;\rho}, \quad \Gamma_{abcd}^{6,2;\rho\rho'} = T_{abcd}^{6,2;\rho\rho'} + \Sigma_{abcd}^{6,2;\rho\rho'}, \quad \Gamma_{abcd}^{4,1;\rho} = T_{abcd}^{4,1;\rho} + \Sigma_{abcd}^{4,1;\rho}. \quad (89)$$

Therefore the equation on figure 9 takes the form

$$\Gamma_{abcd} = \Gamma_{abcd}^{6,1} + \Gamma_{abcd}^{4,1} + \Gamma_{abcd}^{6,2}. \quad (90)$$

All of the above quantities are obtained by using the following symmetry properties:

Proposition 6. • $\Gamma_{abcd}^{6,1;2}$ can be obtained using $\Gamma_{abcd}^{6,1;1}$ and replaced $a \rightarrow b$ and $b \rightarrow a$.

- $\Gamma_{abcd}^{6,1;3}$ is obtained using $\Gamma_{abcd}^{6,1;1}$ and replaced $a \rightarrow c$, $b \rightarrow a$ and $c \rightarrow b$.
 - $\Gamma_{abcd}^{6,1;4}$ is obtained using $\Gamma_{abcd}^{6,1;1}$ and replaced $a \rightarrow d$, $b \rightarrow a$, $c \rightarrow b$ and $d \rightarrow c$.
- This above symmetries is well satisfied for $\Gamma_{abcd}^{4,1;\rho}$. In the case of $\Gamma_{abcd}^{6,2;\rho\rho'}$ we get:

- $\Gamma_{abcd}^{6,2;13}$ can be obtained using $\Gamma_{abcd}^{6,2;14}$ and by replaced $a \rightarrow b$ and $b \rightarrow a$.
- $\Gamma_{abcd}^{6,2;12}$ is obtained by replaced in $\Gamma_{abcd}^{6,2;14}$, $a \rightarrow c$, $b \rightarrow a$ and $c \rightarrow b$.
- $\Gamma_{abcd}^{6,2;23}$ is obtained by replaced in $\Gamma_{abcd}^{6,2;14}$, $a \rightarrow b$, $b \rightarrow a$, $c \rightarrow d$ and $d \rightarrow c$.
- $\Gamma_{abcd}^{6,2;24}$ is obtained by replaced in $\Gamma_{abcd}^{6,2;14}$, $c \rightarrow d$ and $d \rightarrow c$.
- $\Gamma_{abcd}^{6,2;34}$ is obtained by replaced in $\Gamma_{abcd}^{6,2;14}$, $b \rightarrow c$, $c \rightarrow d$ and $d \rightarrow b$.

We then focus our attention to $\Gamma_{abcd}^1 = (\Gamma_{abcd}^{6,1;1} + \Gamma_{abcd}^{4,1;1}) + \Gamma_{abcd}^{6,2;14}$. We also call $G_{[mn]abc}^{ins}$ the two-point functions with insertion (1,2,3) wherein the momentum indices p_1, p_2, p_3 are summed i.e.

$$G_{[mn]abc}^{ins} = \sum_{p_1, p_2, p_3} \langle \varphi_{m123} \bar{\varphi}_{n123} \varphi_{nabc} \bar{\varphi}_{mabc} \rangle_c. \quad (91)$$

The following relations are satisfied:

$$\Sigma_{abcd}^{6,1;1} = Z^2 \lambda_{6,1;1} C_{abcd} \sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins}, \quad T_{abcd}^{6,1;1} = Z^2 \lambda_{6,1;1} C_{abcd} \sum_{p,q,r} G_{pqra}, \quad (92)$$

$$\Sigma_{abcd}^{6,2;14} = Z^2 \lambda_{6,2;14} C_{abcd} \sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins}, \quad T_{abcd}^{6,2;14} = Z^2 \lambda_{6,2;14} C_{abcd} \sum_{p,q,r} G_{pqra}, \quad (93)$$

$$\Sigma_{abcd}^{4,1;1} = Z^2 \lambda_{4,1;1} \sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins}, \quad T_{abcd}^{4,1;1} = Z^2 \lambda_{6,1;1} \sum_{p,q,r} G_{pqra} \quad (94)$$

and then

$$\begin{aligned}\Gamma_{abcd}^1 &= Z^2 C_{abcd} \lambda_{6,1;1} \left[\sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins} + \sum_{p,q,r} G_{pqra} \right] + Z^2 C_{abcd} \lambda_{6,2;14} \left[\sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins} \right. \\ &\quad \left. + \sum_{p,q,r} G_{pqra} \right] + Z^2 \lambda_{4,1;1} \left[\sum_p G_{abcd}^{-1} G_{[ap]bcd}^{ins} + \sum_{p,q,r} G_{pqra} \right].\end{aligned}\quad (95)$$

We set $\lambda_{6,1;\rho} = \lambda_{6,1}$, $\lambda_{6,2;\rho\rho'} = \lambda_{6,2}$ and $\lambda_{4,1;\rho} = \lambda_{4,1}$. Noting that the connected to point function can be expressed as $G_{abcd}^{-1} = M_{abcd} - \Gamma_{abcd}$. Then we get

$$\begin{aligned}\Gamma_{abcd}^1 &= Z^2 M_{abcd}^{-1} \lambda_{6,1} \left[\sum_p G_{abcd}^{-1} \frac{G_{pbcd} - G_{abcd}}{Z(a^2 - p^2)} + \sum_{p,q,r} G_{pqra} \right] \\ &\quad + Z^2 M_{abcd}^{-1} \lambda_{6,2} \left[\sum_p G_{abcd}^{-1} \frac{G_{pbcd} - G_{abcd}}{Z(a^2 - p^2)} + \sum_{p,q,r} G_{pqra} \right] \\ &\quad + Z^2 \lambda_{4,1} \left[\sum_p G_{abcd}^{-1} \frac{G_{pbcd} - G_{abcd}}{Z(a^2 - p^2)} + \sum_{p,q,r} G_{pqra} \right] \\ &= Z^2 M_{abcd}^{-1} \lambda_{6,1} \left[\sum_p \left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}} \frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z(a^2 - p^2)} \right) + \sum_{p,q,r} \frac{1}{M_{pqra} - \Gamma_{pqra}} \right] \\ &\quad + Z^2 M_{abcd}^{-1} \lambda_{6,2} \left[\sum_p \left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}} \frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z(a^2 - p^2)} \right) + \sum_{p,q,r} \frac{1}{M_{pqra} - \Gamma_{pqra}} \right] \\ &\quad + Z^2 \lambda_{4,1} \left[\sum_p \left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}} \frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z(a^2 - p^2)} \right) + \sum_{p,q,r} \frac{1}{M_{pqra} - \Gamma_{pqra}} \right].\end{aligned}\quad (96)$$

Now we use the Taylor expansion that allows to pass to the renormalized quantity as

$$\Gamma_{abcd}^1 = Z m_{bar}^2 - m^2 + (Z - 1)(a^2 + b^2 + c^2 + d^2) + \Gamma_{abcd}^{phys}, \quad (97)$$

with condition $\Gamma_{0000} = 0$ and $\partial\Gamma_{0000} = 0$. This implies that

$$G_{abcd}^{-1} = a^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{abcd}^{ren}. \quad (98)$$

Then we get the following proposition

Proposition 7. *The closed equation of the two-point functions of four dimension tensor model is given by*

$$\begin{aligned}&= M_{abcd}^{-1} \lambda_{6,1} \left\{ \sum_p \left[\frac{(Z - 1)(a^2 + b^2 + c^2 + d^2) + \Gamma_{abcd}^{phys}}{Z} - \frac{1}{m^2} \frac{M_{abcd}}{(p^2 + m^2 - \Gamma_{p000}^{phys})} \right. \right. \\ &\quad \left. \left. - \frac{Z}{p^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{pbcd}^{phys}} \frac{\Gamma_{abcd}^{phys} - \Gamma_{pbcd}^{phys}}{(a^2 - p^2)} + \frac{1}{m^2} \frac{M_{abcd}}{(p^2 + m^2 - \Gamma_{p000}^{phys})} \frac{\Gamma_{p000}^{phys}}{p^2} \right] \right\} \\ &\quad + \sum_{p,q,r} \left[\frac{Z^2}{p^2 + q^2 + r^2 + a^2 + m^2 - \Gamma_{pqra}^{phys}} - \frac{1}{m^2} \frac{Z M_{abcd}}{(p^2 + q^2 + r^2 + m^2 - \Gamma_{pqra}^{phys})} \right] \\ &\quad + M_{abcd}^{-1} \lambda_{6,2} \left\{ \sum_p \left[\frac{Z}{p^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{pbcd}^{phys}} - \frac{1}{m^2} \frac{M_{abcd}}{(p^2 + m^2 - \Gamma_{p000}^{phys})} \right] \right\}\end{aligned}$$

$$\begin{aligned}
& - \frac{Z}{p^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{pbcd}^{phys}} \frac{\Gamma_{abcd}^{phys} - \Gamma_{pbcd}^{phys}}{(a^2 - p^2)} + \frac{1}{m^2} \frac{M_{abcd}}{(p^2 + m^2 - \Gamma_{p000}^{phys})} \frac{\Gamma_{p000}^{phys}}{p^2} \Big] \\
& + \sum_{p,q,r} \left[\frac{Z^2}{p^2 + q^2 + r^2 + a^2 + m^2 - \Gamma_{pqra}^{phys}} - \frac{1}{m^2} \frac{Z M_{abcd}}{(p^2 + q^2 + r^2 + m^2 - \Gamma_{pqr0}^{phys})} \right] \Big\} \\
& + \lambda_{4,1} \left\{ \sum_p \left[\frac{Z}{p^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{pbcd}^{phys}} - \frac{Z}{p^2 + m^2 - \Gamma_{p000}^{phys}} \right] \right. \\
& - \frac{Z}{p^2 + b^2 + c^2 + d^2 + m^2 - \Gamma_{pbcd}^{phys}} \frac{\Gamma_{abcd}^{phys} - \Gamma_{pbcd}^{phys}}{(a^2 - p^2)} + \frac{Z}{p^2 + m^2 - \Gamma_{p000}^{phys}} \frac{\Gamma_{p000}^{phys}}{p^2} \Big] \\
& \left. + \sum_{p,q,r} \left[\frac{Z^2}{p^2 + q^2 + r^2 + a^2 + m^2 - \Gamma_{pqra}^{phys}} - \frac{Z^2}{p^2 + q^2 + r^2 + m^2 - \Gamma_{pqr0}^{phys}} \right] \right\}. \tag{99}
\end{aligned}$$

Proof. The equation (99) can be simply obtained using the relation of $Zm_{bar}^2 - m^2$ in the same way of the last section as

$$\begin{aligned}
Zm_{bar}^2 - m^2 &= ZM_{0000}^{-1} \lambda_{6,1} \left[\sum_p \left(\frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} - \frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} \frac{\Gamma_{p000}^{phys}}{p^2} \right) \right. \\
& + \sum_{p,q,r} \frac{Z}{p^2 + q^2 + r^2 + m^2 - \Gamma_{pqr0}^{phys}} \Big] + ZM_{0000}^{-1} \lambda_{6,2} \left[\sum_p \left(\frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} \right. \right. \\
& - \frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} \frac{\Gamma_{p000}^{phys}}{p^2} \Big) + \sum_{p,q,r} \frac{Z}{p^2 + q^2 + r^2 + m^2 - \Gamma_{pqr0}^{phys}} \Big] + Z\lambda_{4,1} \left[\sum_p \left(\frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} \right. \right. \\
& - \frac{1}{p^2 + m^2 - \Gamma_{p000}^{phys}} \frac{\Gamma_{p000}^{phys}}{p^2} \Big) + \sum_{p,q,r} \frac{Z}{p^2 + q^2 + r^2 + m^2 - \Gamma_{pqr0}^{phys}} \Big], \quad M_{0000}^{-1} = \frac{1}{Zm^2} \tag{100}
\end{aligned}$$

Then (99) takes the form by replacing the relation (100) into the right hand side of equation (96). \square

Let us remark that the continuous limit of the equation (99) can be built. We identify the sum as $\sum_p = 2 \int_0^\infty dp$ and $\sum_{p,q,r} = 2 \int_0^\infty p^2 dp$. We also impose the cutoff p_Λ in the UV and changing the variables as

$$\begin{aligned}
a^2 &= m^2 \frac{\alpha}{1 - \alpha}, \quad b^2 = m^2 \frac{\beta}{1 - \beta}, \quad c^2 = m^2 \frac{\gamma}{1 - \gamma}, \\
d^2 &= m^2 \frac{\epsilon}{1 - \epsilon}, \quad p^2 = m^2 \frac{\rho}{1 - \rho}, \quad p_\Lambda^2 = m^2 \frac{\Lambda}{1 - \Lambda}. \tag{101}
\end{aligned}$$

Now let us define the two quantities $s(\alpha, \beta, \gamma, \epsilon)$ and $p(\alpha, \beta, \gamma, \epsilon)$ as

$$s(\alpha, \beta, \gamma, \epsilon) = 1 - \alpha\beta - \alpha\gamma - \alpha\epsilon - \beta\gamma - \beta\epsilon - \gamma\epsilon + 2\alpha\beta\gamma + 2\alpha\beta\epsilon + 2\alpha\gamma\epsilon + 2\beta\gamma\epsilon - 3\alpha\beta\gamma\epsilon \tag{102}$$

and

$$p(\alpha, \beta, \gamma, \epsilon) = (1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \epsilon). \tag{103}$$

The equation (99) is re-expressed as

$$m^2(Z - 1) \frac{p(\alpha, \beta, \gamma, \epsilon)}{(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \epsilon)} + m^2 \frac{\Gamma_{\alpha\beta\gamma\epsilon}}{(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \epsilon)}$$

$$\begin{aligned}
&= 2M_{\alpha\beta\gamma\epsilon}^{-1}(\lambda_{6,1} + \lambda_{6,2}) \left\{ \int_0^\Lambda \frac{1}{2m} \sqrt{\frac{1-\rho}{m\rho}} \frac{d\rho}{(1-\rho)^2} \left[\frac{Z(1-\rho)(1-\beta)(1-\gamma)(1-\epsilon)}{s(\rho, \beta, \gamma, \epsilon) - \Gamma_{\rho\beta\gamma\epsilon}} \right. \right. \\
&- \frac{1}{m^2} \frac{M_{\alpha\beta\gamma\epsilon}(1-\rho)}{1 - \Gamma_{\rho 000}} - \frac{Z(1-\rho)}{(s(\rho, \beta, \gamma, \epsilon) - \Gamma_{\rho\beta\gamma\epsilon})} \frac{(\alpha - \rho)}{(1-\rho)\Gamma_{\alpha\beta\gamma\epsilon} - (1-\alpha)\Gamma_{\rho\beta\gamma\epsilon}} \\
&+ \left. \frac{1}{m^2} \frac{M_{\alpha\beta\gamma\epsilon}(1-\rho)}{(1 - \Gamma_{\rho 000})} \frac{\Gamma_{\rho 000}}{\rho} \right] + \int_0^\Lambda \frac{1}{2m} \sqrt{\frac{m\rho}{1-\rho}} \frac{d\rho}{(1-\rho)^2} \left[\frac{Z^2(1-\rho)^3(1-\alpha)}{s(\rho, \rho, \rho, \alpha) - \Gamma_{\rho\rho\rho\alpha}} \right. \\
&- \left. \frac{1}{m^2} \frac{Z M_{\alpha\beta\gamma\epsilon}(1-\rho)^3}{(2\rho^3 - 3\rho^2 + 1 - \Gamma_{\rho\rho\rho 0})} \right] \Big\} \\
&+ \lambda_{4,1} \left\{ \int_0^\Lambda \frac{1}{2m} \sqrt{\frac{1-\rho}{m\rho}} \frac{d\rho}{(1-\rho)^2} \left[\frac{Z(1-\rho)(1-\beta)(1-\gamma)(1-\epsilon)}{s(\rho, \beta, \gamma, \epsilon) - \Gamma_{\rho\beta\gamma\epsilon}} \right. \right. \\
&- \frac{Z(1-\rho)}{1 - \Gamma_{\rho 000}} - \frac{Z(1-\rho)}{(s(\rho, \beta, \gamma, \epsilon) - \Gamma_{\rho\beta\gamma\epsilon})} \frac{(\alpha - \rho)}{(1-\rho)\Gamma_{\alpha\beta\gamma\epsilon} - (1-\alpha)\Gamma_{\rho\beta\gamma\epsilon}} \\
&+ \left. \frac{Z(1-\rho)}{(1 - \Gamma_{\rho 000})} \frac{\Gamma_{\rho 000}}{\rho} \right] + \int_0^\Lambda \frac{1}{2m} \sqrt{\frac{m\rho}{1-\rho}} \frac{d\rho}{(1-\rho)^2} \left[\frac{Z^2(1-\rho)^3(1-\alpha)}{s(\rho, \rho, \rho, \alpha) - \Gamma_{\rho\rho\rho\alpha}} \right. \\
&- \left. \frac{Z^2(1-\rho)^3}{(2\rho^3 - 3\rho^2 + 1 - \Gamma_{\rho\rho\rho 0})} \right] \Big\}. \tag{104}
\end{aligned}$$

The wave function Z can be also deduced as

$$Z = \frac{1 - \frac{2}{m^2}(\lambda_{6,1} + \lambda_{6,2}) \int_0^\Lambda \frac{d\rho}{2m^3} \sqrt{\frac{1-\rho}{m\rho}} \left(G_{\rho 000} + \frac{G'_{\rho 000}}{\rho} \right)}{1 + \lambda_{4,1} \int_0^\Lambda \frac{d\rho}{2m^3} \sqrt{\frac{1-\rho}{m\rho}} \left(G_{\rho 000} + \frac{G'_{\rho 000}}{\rho} \right)}, \tag{105}$$

where

$$s(\alpha, \beta, \gamma, \epsilon) - \Gamma_{\alpha\beta\gamma\epsilon} = \frac{s(\alpha, \beta, \gamma, \epsilon)}{G_{\alpha\beta\gamma\epsilon}}, \tag{106}$$

and

$$M_{\alpha\beta\gamma\epsilon} = Z m^2 \frac{s(\alpha, \beta\gamma, \epsilon)}{p(\alpha, \beta\gamma, \epsilon)}. \tag{107}$$

Finally by replacing the expressions (105) and (106) in the equation (104), we obtain the closed equation in the continuous limit, which will also be fully addressed in forthcoming work.

5 Conclusion

In the present paper, we have presented a perturbative calculation of two-point correlation functions of rank 3 TGFT. As discussed earlier the correlation functions are given by combining Ward-Takahashi identities and Schwinger-Dyson equations that allows to establish the appropriate closed equation. The closed equation in the $4D$ case is also given.

In this work, we proved that the nonperturbative techniques as developed in [29][30][28] can be reported to the tensor situation. Indeed, although, we only solve our closed form equations for the two-point functions at initial orders, it is very promising to see that we can obtain even solutions in this highly combinatoric case. As future investigations, we can now undertake a calculation of the general solution at all orders of the coupling constants for both rank 3 and 4 models.

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